

# TECHNICAL NOTE

D-1224

COMBINATIONS OF TEMPERATURE AND  
AXIAL COMPRESSION REQUIRED FOR BUCKLING  
OF A RING-STIFFENED CYLINDER

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## SUMMARY

A theory is presented to predict the buckling temperature of an axially compressed, uniformly heated ring-stiffened cylinder. The cylinder buckles because of the interaction of the axial stress due to applied compressive loads and the circumferential stress resulting from restraint of thermal expansion by the rings. Buckling charts covering a wide range of cylinder proportions are presented for both clamped and simply supported cylinders. The buckling temperature for a given axial loading is determined from a simple equation involving a coefficient given in the buckling charts and the radius-thickness ratio of the cylinder.

## INTRODUCTION

One aspect of the structural-aerodynamic heating problem is the effect of thermal stresses on buckling. Reference 1 reports the results of an experimental investigation which indicated the effect of thermal stresses on buckling and maximum strength for a variety of structures. The results for the ring-stiffened cylinders indicated the compressive buckling stress was reduced by thermal stresses. These thermal stresses were in the circumferential direction and caused by the restraint of thermal expansion in the vicinity of the rings.

Reference 2 presented a theoretical analysis of the buckling of a simply supported cylinder due to this type of thermal stress alone. The calculated buckling temperature of a steel cylinder with a radius-thickness ratio of about 300 was found to be over 2,300° F; thus, there appeared to be no problem in the practical temperature range of the material. Calculations for other proportions were not given. For larger values of radius-thickness ratio, the buckling temperature is lower and, as shown later, thermal buckling can occur for cylinders of practical proportions. The present paper presents a theoretical analysis

for buckling of a cylinder subject to axial compression as well as thermal stress for both simply supported and clamped end conditions. A comparison of the theory with the experimental results of reference 1 is also given.

## SYMBOLS

$A_R$	cross-sectional area of ring	
$a_m$	coefficient in deflection function	I
$B$	constant (see eq. (6))	1
$D$	plate flexural stiffness, $\frac{Et^3}{12(1 - \mu^2)}$	9
$E$	modulus of elasticity	6
$k$	circumferential stress coefficient for arbitrary stress distribution	C
$k_T$	temperature coefficient, $\frac{12(\alpha T - \alpha_R T_R)(1 - \mu^2)}{\pi^2} \left(\frac{L}{t}\right)^2$	
$k_T'$	modified temperature coefficient applicable to cylinders with flexible rings	
$k_x$	axial stress coefficient, $\frac{12\sigma_x(1 - \mu^2)}{\pi^2 E} \left(\frac{L}{t}\right)^2$	
$k_y$	circumferential stress coefficient for uniform stress distribution, $\frac{12\sigma_y(1 - \mu)^2}{\pi^2 E} \left(\frac{L}{t}\right)^2$	
$L$	length of cylinder between rings	
$M_m$	$= \left[ (m - 1)^2 + \beta^2 \right]^2 + \frac{12Z^2}{\pi^4} \frac{(m - 1)^4}{\left[ (m - 1)^2 + \beta^2 \right]^2} - (m - 1)^2 k_x$	
$m, n, i$	integers	

Q	operator defined in equation (A5)
r	radius of cylinder
$s_n$	Fourier components of circumferential stress
T	temperature
$\Delta T$	temperature difference between cylinder wall and ring, $T - T_R$
t	thickness of cylinder wall
$V_m$	deflection function
w	radial deflection of cylinder wall
x,y	axial and circumferential coordinates (see fig. 12)
Z	cylinder curvature parameter, $\frac{L^2}{rt} \sqrt{1 - \mu^2}$
$\alpha$	coefficient of thermal expansion
$\beta$	ratio of cylinder length to circumferential buckle length, $L/\lambda$
$\gamma$	axial stress ratio; ratio of axial stress to the classical cylinder buckling stress, $\frac{\sigma_x}{\frac{Et}{\sqrt{3(1 - \mu^2)}r}}$
$\gamma'$	modified axial stress ratio based on experimentally observed reduction in cylinder buckling stress from classical value
$\delta_{mn}$	Kronecker delta, $\delta_{mn} = 0$ when $m \neq n$ and $\delta_{mn} = 1$ when $m = n$
$\theta$	$\theta = \sqrt{\sqrt{3} Z} \sqrt{1 - \gamma}$
$\phi$	$\phi = \sqrt{\sqrt{3} Z} \sqrt{1 + \gamma}$
$\lambda$	circumferential buckle length
$\mu$	Poisson's ratio ( $\mu$ equal 0.3 was used for all numerical calculations)

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$\sigma_x$	axial compressive stress
$\sigma_y$	circumferential compressive stress
$\bar{\sigma}_y$	average circumferential compressive stress
$\tau$	temperature coefficient, $(\alpha T - \alpha_R T_R) \frac{F}{t}$
$\tau'$	modified temperature coefficient applicable to cylinders with flexible rings
$\psi$	deflectional stiffness of ring in radial direction
$\nabla^4$	differential operator, $\frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$
$\nabla^{-4}$	inverse operator defined such that $\nabla^4 \nabla^{-4} w = w$

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Subscripts:

max	maximum
R	refers to ring
0	evaluated at $\tau$ equal zero

Bars over symbols indicate average values.

## THEORY AND ASSUMPTIONS

The cylinder is assumed to have many rings which divide the cylinder into bays of length  $L$ . The cylinder wall is at a uniform temperature and the rings are at some lower temperature. This difference in temperature causes circumferential stresses which vary in the axial direction and have the largest magnitudes in the vicinity of the rings. Combinations of axial compression and temperature (circumferential thermal stress) necessary to cause buckling were determined from the analyses presented in the appendixes.

In appendix A, the problem considered is buckling of a cylinder subjected to a uniform axial stress and a circumferential stress that varies in the axial direction and is expressed as a general Fourier

series. In appendix B, the circumferential thermal stress resulting from the temperature difference between the cylinder wall and rings is determined; this result is expressed as a Fourier series in order to be applicable to the buckling analysis of appendix A. One of the major assumptions of the whole analysis is in applying the thermal stress results to the buckling problem. It is assumed that circumferential stresses produced by the temperature rise of the cylinder can be used in the buckling analysis. However, the deflections caused by the temperature rise are neglected inasmuch as a perfect cylinder is assumed in the buckling analysis. Some support of this assumption is given by the fact that these deflections are generally small and not compatible with the final buckle pattern.

In the stability analysis of appendix A, the modified equation of equilibrium presented by Batdorf (ref. 3) was used. The buckle deflection was expressed as a trigonometric series and an infinite stability determinant resulted when the series was substituted into the equilibrium equation. For simply supported cylinders, the results obtained using the modified equation are the same as those that would be obtained by using Donnell's eighth-order differential equation. However, as pointed out in reference 3, the use of Donnell's equation for clamped cylinders may lead to divergent trigonometric series but this problem is not encountered with the use of the modified equation.

The usual assumptions of small-deflection cylinder theory together with the assumptions implied by the use of the modified equilibrium equation are present in the analysis. These assumptions include certain restrictions on the in-plane displacements at the end of the cylinder. For the deflection functions used in this paper, the following conditions are implied on the in-plane displacements at the ends of the cylinder: for simply supported cylinders, free displacement in the axial direction and zero displacement in the circumferential direction; and for clamped cylinders, zero displacement in the axial direction and free displacement in the circumferential direction. The condition of free displacement in the circumferential direction for a clamped cylinder does not seem to be too realistic but it is pointed out in reference 3 that there is little difference in critical stress obtained with this boundary condition and the critical stress obtained by other investigators using a boundary condition that all edge displacements were zero.

The rings are assumed to be rigid against radial loads but are allowed to expand because of a temperature rise. A correction which allows for the ring expansion that occurs when the cylinder is heated and loaded is given in a later section.

## THEORETICAL RESULTS

The results of the analysis are presented in figures 1 and 2 and in tabular form in table I. Results are presented as a buckling temperature coefficient  $\tau$  plotted against the cylinder curvature parameter  $Z$  for various values of  $\gamma$ , the ratio of applied axial stress to the so-called classical buckling stress. The buckling temperature can be calculated as

$$T = \frac{\alpha_R}{\alpha} T_R + \frac{\tau}{\alpha} \frac{t}{r} \quad (1)$$

or, if the coefficient of thermal expansion of ring and skin can be considered equal ( $\frac{\alpha_R}{\alpha} = 1$ ), then

$$\Delta T = \frac{\tau}{\alpha} \frac{t}{r} \quad (2)$$

In practice,  $T$  will represent the average skin temperature,  $T_R$  the average ring temperature, and  $\Delta T$  the difference between these average temperatures. The values of  $\tau$  shown are the lower of the two values found for the symmetrical and antisymmetrical deflection patterns and have been minimized with respect to the wavelength parameter  $\beta$ , the ratio of the cylinder length to circumferential buckle length. Although the precise minimum was not found, the increments in  $\beta$  were suitably small so that little error is present in using the lowest value that was calculated.

Inasmuch as the thermal stress is concentrated over the rings, many terms in the deflection function are required to describe the buckle pattern accurately, especially at large values of  $Z$ . Calculations were made on an IBM 7090 electronic data processing system and by increasing the order of the stability determinant suitable accuracy was achieved except in some cases as indicated later. The results given are such that using a determinant of one less order would yield a value of  $\tau$  no more than 5 percent greater. However, except for the larger values of  $Z$ , most of the values of  $\tau$  given in the table differ by less than 1 percent from values obtained by using a determinant of one less order. The order of determinant used as well as the value of  $\beta$  is given in the table. In some cases it was not possible to get an accurate value of  $\tau$  as the size of the determinant was limited by the computer program. These points are omitted from the table but extrapolations based on established calculations are shown by dashed lines in the figures. At



low values of  $Z$  where buckling is associated with plate action rather than cylinder action, it is possible for  $\gamma$  to be greater than 1 and calculations for these points are given in the table only. The results given in figures 1 and 2 show that, as  $Z$  is increased,  $\gamma$  being held constant,  $\tau$  becomes independent of  $Z$ . This is true for all values of  $\gamma$  for the simply supported cylinder. However, it is not completely definite that this is true for the smaller values of  $\gamma$  in the case of the clamped cylinder because of the difficulty in obtaining an accurate value of  $\tau$ . Although values are given only to  $Z = 1,000$ , calculations for  $Z$  greater than 1,000 indicate the trend shown would continue.

The value of  $\tau$  for  $\gamma$  equal to zero can be used to determine the temperature at which a cylinder will buckle in the absence of axial stress. For the range of  $Z$  in which  $\tau$  is independent of  $Z$ , the buckling temperature is a function of  $r/t$  and the coefficient of thermal expansion  $\alpha$ , as indicated in figure 3 where the temperature rise necessary to cause buckling of aluminum and stainless-steel cylinders is plotted against  $r/t$ . A constant value of  $\alpha$  was assumed for ring and skin in each case. At low values of  $r/t$  the buckling temperature is very high and is beyond the useful temperature range for the material. However, at high values of  $r/t$  the buckling temperature can be low enough that thermal buckling would have to be considered in the design of a heated cylinder. It should be noted that the curves in figure 3 also apply to values of  $\gamma$  up to 0.7.

The buckling temperature of a simply supported cylinder subjected to heating alone can also be obtained from the analysis of reference 2. In the numerical example of reference 2, a buckling temperature rise of  $2,380^\circ \text{F}$  was calculated for a steel cylinder with  $r/t$  equal to 300. It can be seen that this temperature is about three times the value shown in figure 3. The difference is probably due to the boundary conditions used in calculating the thermal stress as the other parts of the two analyses are identical. Reference 2 assumed a one-bay cylinder and calculated the thermal stress for simply supported ends. The present analysis assumes many bays; this assumption effectively clamps the ends as far as thermal stress is concerned. (See appendix B.)

For the range of  $Z$  in which  $\tau$  is independent of  $Z$ , the results given in figures 1 and 2 can be cross plotted to give curves for the variation of  $\tau$  with  $\gamma$  as shown in figure 4. As expected, when axial compression is the dominant factor in buckling, there is little difference between a clamped and simply supported cylinder. However, at small values of  $\gamma$ , circumferential stress is the most important factor in buckling and there is a significant difference between the results for a clamped cylinder and the results for a simply supported cylinder. This difference is consistent with that obtained for uniform circumferential compression.

The curves of figure 4 indicate that an appreciable amount of compression can be applied without reducing the temperature at which the cylinder will buckle. For the clamped cylinder, the buckling temperature can actually increase with applied load. This result appears to be contrary to intuition but it can be shown that the average circumferential stress is declining even though the temperature is increasing.

It is of interest to note that in figure 4  $\tau$  is zero for  $\gamma$  approximately equal to 0.95. This result means that the theoretical buckling stress for an unheated cylinder ( $\tau = 0$ ) is about 95 percent of the classical value because of the restraint of Poisson's expansion at the rings or bulkheads.

The sharp break in the curves of figure 4 is the result of a mode change which is illustrated in figure 5. The coefficient  $\tau$  is plotted against  $\beta$  for a clamped cylinder with  $Z$  equal to 50 and  $\gamma$  equal to 0.6 and 0.7. For each value of  $\gamma$ , curves corresponding to the two lowest buckling modes are shown. The controlling mode for  $\gamma$  equal to 0.6 has a minimum value of  $\tau$  at  $\beta$  equal to 5.4. For  $\gamma$  equal 0.7 there is another minimum in the same region; however, there is a lower minimum at  $\beta$  equal to 2.4. It can be seen that a mode change has taken place and that a sharp break will occur in a plot of the minimum value of  $\tau$  against  $\gamma$  when going from one mode to the other. At the lower values of  $\gamma$ , where buckling is due primarily to circumferential stress, the  $\tau - \beta$  curves for the controlling mode shape are characterized by rather flat minimums and require large stability determinants for convergence. Calculations of the associated buckle deflection indicate large deflections near the ends where the thermal stress is concentrated and essentially zero deflection away from the ends. At larger values of  $\gamma$ , the mode shape changes, and the  $\tau - \beta$  curves have sharper minimums and do not require as large a stability determinant as the lower values of  $\gamma$ . For this case the calculated buckle deflections are in a fairly regular wave pattern over the entire length of the cylinder as is the case for compressive loads acting alone.

#### Effect of Thermal Stress Distribution on Buckling

As mentioned previously, the thermal stress is concentrated in the vicinity of the rings as indicated in figure 6. The circumferential stress at a ring ( $x = 0$ ) can be obtained from equations (A3) and (B5) as

$$\sigma_y(0) = \mu\sigma_x + \alpha ET - \alpha_R E T_R \quad (3)$$

The circumferential stress distribution was obtained from appendix B and is shown for two values of  $Z$  and two values of  $\gamma$  in figure 6 as the ratio of circumferential stress to circumferential stress at a ring plotted against  $x/L$ . A study of figure 6 and additional thermal stress calculations indicate that for cylinders of a given radius and thickness the thermal stress distribution in the vicinity of the rings is unaffected by increases in length beyond a certain value, and the thermal stress away from the rings is essentially zero for these "long" cylinders. This behavior is probably the reason that the buckling temperature is independent of length for the larger values of  $Z$ .

The average value of the circumferential stress  $\bar{\sigma}_y$  is shown in figure 6 as the line denoted  $\frac{\bar{\sigma}_y}{\sigma_y(0)}$ . It is of interest to compare the average value of the thermal buckling stress with the uniform circumferential stress that causes buckling. Values of  $k_y$ , the buckling coefficient for uniform circumferential compression, can be obtained as a special case from the analysis in appendix A. Inasmuch as the thermal stress over the ring can be obtained from the buckling coefficient  $k_T$ , a buckling coefficient corresponding to the average thermal stress is  $\frac{\bar{\sigma}_y}{\sigma_y(0)}k_T$ . A comparison of the buckling coefficient for uniform circumferential compression  $k_y$  with the buckling coefficient for the average thermal stress  $\frac{\bar{\sigma}_y}{\sigma_y(0)}k_T$  is given in figure 7. A buckling coefficient based on cylinder length was used; this manner of presentation corresponds to the way in which results are usually presented for uniform circumferential compression.

At low values of  $Z$  the thermal stress distribution is almost uniform so  $\frac{\bar{\sigma}_y}{\sigma_y(0)}k_T$  and  $k_y$  are almost identical and approach the value for a flat plate as  $Z$  approaches zero. As  $Z$  increases, the thermal stress distribution becomes similar to that shown in figure 6 and it can be seen from figure 7 that the average thermal stress at buckling is about three to four times the buckling stress for a uniform distribution for both simply supported and clamped cylinders.

#### Effect of Ring Flexibility on Buckling

The results given in table I and figures 1 and 2 apply to rings which do not deflect radially when the skin is heated or loaded in axial

compression. The deflection an actual ring undergoes can be taken into account as indicated in appendix B. The buckling temperature coefficient  $\tau'$  for nonrigid rings is given in terms of the rigid-ring solution plus an additional term involving the average circumferential stress in the skin and the ratio of skin and ring stiffnesses. The coefficient  $\tau'$  is given as

$$\tau' = \tau + \left( \frac{\mu\gamma}{\sqrt{3}(1 - \mu^2)} + \tau \right) \frac{\bar{\sigma}_y}{\sigma_y(0)} \frac{ELt}{E_R A_R} \quad (4)$$

It may be necessary in applying equation (4) to use an effective ring area  $A_R$  if the entire ring does not fully participate in resisting the

expansion of the skin. Values of  $\frac{\bar{\sigma}_y}{\sigma_y(0)}$  which are necessary for the use of equation (4) have been calculated and are shown in figure 8 plotted against  $Z$ . It is seen that for  $Z$  small,  $\frac{\bar{\sigma}_y}{\sigma_y(0)}$  approaches 1 because the stress distribution is nearly uniform. As  $Z$  increases,  $\frac{\bar{\sigma}_y}{\sigma_y(0)}$  becomes proportional to  $1/\sqrt{Z}$  which can be considered as proportional to  $1/L$  for  $r/t$  constant. Such a result conforms to the observations made in the previous section, that the thermal stresses are concentrated near the rings and are independent of cylinder length for the larger values of  $Z$ . Thus for a given radius and thickness, the net compression force resulting from the thermal stress is constant and the average value of thermal stress is proportional to  $1/L$  as indicated in figure 8. The oscillations of  $\frac{\bar{\sigma}_y}{\sigma_y(0)}$  at higher values of  $\gamma$  are about one-half cycle out of phase with the oscillations of  $\tau$  in figures 1 and 2. Thus,  $\tau \frac{\bar{\sigma}_y}{\sigma_y(0)}$ , which may be thought of as an average stress buckling coefficient, is a smoother function of  $Z$  than  $\tau$ . In the region where  $\frac{\bar{\sigma}_y}{\sigma_y(0)}$  is proportional to  $1/\sqrt{Z}$ , equation (4) can be written independent of length for intermediate to large values of  $Z$  as

$$\tau' = \tau + \left( \frac{\mu\gamma}{\sqrt{3}(1 - \mu^2)} + \tau \right) \frac{2\sqrt{1 - \gamma}}{\sqrt{3}(1 - \mu^2)} \sqrt{\frac{rt^3}{A_R}} \frac{E}{E_R} \quad (5)$$

## APPLICATION OF RESULTS AND COMPARISON WITH EXPERIMENT

## Modified Theory

Because of the discrepancy between experimental results and small-deflection buckling theory for cylinders under axial compression the present theory cannot be expected to predict experimental results accurately for cases in which axial compression is the predominant factor in buckling. It is necessary therefore to modify the results similar to what has been done for cylinders under compression. Of course, any modification should be such that experimental data for cylinders under uniform compression agree with the modified theory when  $\tau$  equals zero. Reference 4 presents an empirical approach to obtain a reduced buckling stress for cylinders under uniform compression. The buckling stress is given by

$$\sigma_x = \sqrt{\frac{1}{1 + 6B\frac{r}{t}}} \frac{E}{\sqrt{3(1 - \mu^2)}} \frac{t}{r} \quad (6)$$

In equation (6), B is an experimentally determined constant which may be thought of as a measure of initial imperfections present in the cylinder wall; if B is zero, equation (6) is the usual small-deflection buckling equation. If a prime is used to indicate a modified buckling ratio and a subscript zero used to indicate  $\tau$  being equal to zero, equation (6) can be written

$$\gamma'_0 = \sqrt{\frac{1}{1 + 6B\frac{r}{t}}} \quad (7)$$

A simple extension of this equation that will give a reduced value of  $\gamma$  for any value of  $\tau$  is

$$\gamma' = \frac{\gamma}{\gamma'_0} \sqrt{\frac{1}{1 + 6B\frac{r}{t}}} \quad (8)$$

From the shape of the  $\tau - \gamma$  interaction curve given in figure 4 it can be seen that equation (8) will provide the proper amount of correction in

the vicinity of  $\tau$  equal zero but, at lower values of  $\gamma$  where the curve is relatively flat and the circumferential stress is the predominant factor in buckling, little change in the interaction curve will be obtained. Inasmuch as the buckling stress of a cylinder loaded in circumferential compression agrees fairly well with theory, a correction should not be required where circumferential compression is the major factor in buckling. Therefore, equation (8) may be expected to give a reasonable estimate of the compressive buckling coefficient when a cylinder is heated despite the limited amount of experimental data presently available to substantiate such a correction. However, some judgment must be exercised in applying equation (8) dependent on the shape of the interaction curve. For example, in a region of the curve where  $\tau$  increases with  $\gamma$ , as occurs for clamped cylinders, the type of correction required is not clear.

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### Comparison of Theory and Experiment

In reference 1, buckling test results are given for several 2024-T3 aluminum cylinders loaded in bending and uniformly heated. Although the theory of appendix A applies to uniform compression only, it is assumed that bending tests can be compared directly with the theory for uniform compression. Theoretical studies such as reference 5 have indicated that the maximum compressive stress at buckling for a cylinder loaded in pure bending is essentially the classical buckling stress for uniform compression. Experimental results have shown a small difference between compression and bending (see ref. 5), but this is accounted for by the choice of  $B$  in equation (8). A comparison of the results of reference 1 with the theory of the present paper, including the reduction in axial buckling stress indicated in equation (8), is shown in figure 9. The extreme fiber compressive stress at buckling is plotted against the maximum cylinder temperature. The theoretical curves were calculated by assuming simply supported ends, although assuming clamped ends would yield essentially the same results for the range of variables encountered. The correction for ring flexibility (eq. (3)) was used but did not alter the results appreciably from the rigid-ring case.

In order to calculate the curves shown in figure 9, it is necessary to know the variation of the physical quantities  $E$  and  $\alpha$  with temperature as well as the relationship of the skin and ring average temperatures to the maximum cylinder temperature. The values of  $\alpha$  and  $E$  that were used in the calculations are shown in figure 10. The difference in expansion coefficient for the skin and the lower temperature ring was neglected; therefore, only the difference in average ring and skin temperature is required which is given in figure 11 as a function of maximum cylinder temperature. It was not necessary to use equation (8) to modify the theory for the cylinder having  $L/r$  equal to  $1/4$  inasmuch as little difference between small-deflection theory and experiment was found for

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this proportion as indicated by the room-temperature test point. For the remaining cylinders a value of  $B$  equal 0.00030 was used in equation (8). It was found in reference 4 that this value gives results which are in the middle of the experimental scatter band for 7075-T6 aluminum cylinders. The point for  $L/r$  equal to 1 which is below the theory would lie above the theory if the lower limit of the scatter band of the data of reference 4 were used to establish  $B$ . Thus, it appears that the buckling theory of the present paper including the modification given in equation (8) will agree with experimental results about as well as the methods available to predict buckling at room temperature agree with room-temperature buckling results. A better evaluation of the theory could be obtained from cylinders with large values of  $r/t$  so that the buckling temperature would be lower and in a more usable temperature range for the material.

### CONCLUSIONS

The buckling temperature of a ring-stiffened cylinder loaded in compression is shown to be a function of the radius-thickness ratio of the cylinder, the thermal expansion coefficient of the cylinder material, and a buckling coefficient which is presented in tabular and graphical form for both clamped and simply supported cylinders. The buckling coefficient is a function of the ratio of applied axial stress to the classical buckling stress and the cylinder curvature parameter. From the results of the analysis, the following conclusions can be drawn:

1. For moderate to large values of the curvature parameter, the cylinder buckling temperature is essentially independent of length.

2. For small values of the radius-thickness ratio buckling of the cylinder due to temperature alone occurs beyond the useful temperature range for most materials. For larger values of the radius-thickness ratio such buckling can occur at temperatures within the usable temperature range for the material.

3. A cylinder can support a significant amount of axial load without reducing the buckling temperature.

4. For moderate to large values of the curvature parameter, there is little difference between the buckling temperatures for clamped and simply supported cylinders of the same proportions when the axial stress approaches the classical buckling stress. At lower values of axial stress there is a substantial difference.

5. The theoretical buckling stress for an unheated cylinder under uniform axial compression is about 95 percent of the classical value due to the restraint of Poisson expansion at the ends.

A modification of the theory is given which takes into account the reduction in buckling stress from the classical value for cylinders under axial compression. Comparison of the modified theory with experiment indicates agreement comparable to that obtained by using existing empirical methods to predict room-temperature compressive buckling results.

Langley Research Center,  
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Langley Air Force Base, Va., Jan. 15, 1962.

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## APPENDIX A

BUCKLING OF A CIRCULAR CYLINDER UNDER UNIFORM AXIAL COMPRESSION  
AND AN ARBITRARY DISTRIBUTION OF CIRCUMFERENTIAL STRESS

The equation of equilibrium governing the buckling of a cylindrical shell subjected to in-plane direct stresses only is (see ref. 3)

$$D \nabla^4 w + \frac{Et}{r^2} \nabla^{-4} \frac{\partial^4 w}{\partial x^4} + \sigma_x t \frac{\partial^2 w}{\partial x^2} + \sigma_y t \frac{\partial^2 w}{\partial y^2} = 0 \quad (A1)$$

Dividing equation (A1) by  $D$  gives

$$\nabla^4 w + 12 \frac{z^2}{L^4} \nabla^{-4} \frac{\partial^4 w}{\partial x^4} + k_x \frac{\pi^2}{L^2} \frac{\partial^2 w}{\partial x^2} + k \frac{\pi^2}{L^2} \frac{\partial^2 w}{\partial y^2} \sum_{n=0}^{\infty} \frac{s_n}{1 + \delta_{0n}} \cos \frac{n\pi x}{L} = 0 \quad (A2)$$

where

$$z = \frac{L^2}{rt} \sqrt{1 - \mu^2}$$

$$\sigma_x = \frac{k_x \pi^2 D}{L^2 t}$$

and the arbitrary distribution of  $\sigma_y$  is given by the Fourier series

$$\sigma_y = \frac{k \pi^2 D}{L^2 t} \sum_{n=0}^{\infty} \frac{s_n}{1 + \delta_{0n}} \cos \frac{n\pi x}{L}$$

The  $s_n$  values are defined so that  $\sigma_y$  at  $x$  equal zero is given by

$$\sigma_y(0) = \frac{k \pi^2 D}{L^2 t} \quad (A3)$$

This definition requires that

$$\sum_{n=0}^{\infty} \frac{s_n}{1 + \delta_{0n}} = 1 \quad (A4)$$

It can then be seen that  $\frac{s_n}{1 + \delta_{0n}}$  is the ratio of the amplitude of the  $n$ th component of stress to the stress at  $x$  equal zero.

The method used for the solution of equation (A2) is the Galerkin method. (See ref. 3.) In this method the differential equation of equilibrium is represented as

$$Q(w) = 0 \quad (A5)$$

where in equation (A2)

$$Q = \nabla^4 + \frac{12Z^2}{L^4} \nabla^{-4} \frac{\partial^4}{\partial x^4} + k_x \frac{\pi^2}{L^2} \frac{\partial^2}{\partial x^2} + k \frac{\pi^2}{L^2} \frac{\partial^2}{\partial y^2} \sum \frac{s_n}{1 + \delta_{0n}} \cos \frac{n\pi x}{L}$$

The deflection  $w$  is taken as

$$w = \sum_{m=1}^{\infty} a_m V_m$$

The set of functions  $V_m$  must satisfy boundary conditions but not necessarily the equilibrium equation. The coefficients  $a_m$  are determined from the condition

$$\int_0^{\lambda} \int_0^L V_m Q(w) dx dy = 0 \quad (A6)$$

The details of the solution follow.

## Simply Supported Ends

The orientation of the coordinate system with respect to the cylinder is shown in figure 12. An expression for  $w$  which satisfies the differential equation in the  $y$ -direction and the condition of simple support at the ends of each bay is

$$w = \sin \frac{\pi y}{\lambda} \sum_{m=1}^{\infty} a_m \sin \frac{m\pi x}{L} \quad (A7)$$

and  $V_m$  can be identified as

$$V_m = \sin \frac{\pi y}{\lambda} \sin \frac{m\pi x}{L} \quad (A8)$$

Substitution of equations (A7) and (A8) into equation (A6) and integrating over the limits yields

$$a_m M_{m+1} - \frac{k\beta^2}{2} \sum_{i=1}^{\infty} a_i (s_{m-i} - s_{m+i}) = 0 \quad (m = 1, 2, 3, \dots) \quad (A9)$$

where by definition

$$s_{-i} = s_i$$

and

$$M_m = \left[ (m-1)^2 + \beta^2 \right]^2 + \frac{12Z^2(m-1)^4}{\pi^4 \left[ (m-1)^2 + \beta^2 \right]^2} - (m-1)^2 k_x$$

The condition necessary for buckling is that the determinant of the coefficients of equations (A9) equal zero. This determinant can be written as

$$\begin{array}{ccccccc}
& a_1 & a_2 & a_3 & \dots & & \\
m = 1 & \left| \begin{array}{c} \frac{2M_2}{\beta^2 k} - s_0 + s_2 \\ -s_1 + s_3 \\ -s_2 + s_4 \end{array} \right. & \left| \begin{array}{c} -s_1 + s_3 \\ \frac{2M_3}{\beta^2 k} - s_0 + s_4 \\ -s_1 + s_5 \end{array} \right. & \left| \begin{array}{c} -s_2 + s_4 \\ -s_1 + s_5 \\ \frac{2M_4}{\beta^2 k} - s_0 + s_6 \end{array} \right. & \left| \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \right. & = 0 & (A10) \\
m = 2 & & & & & & \\
m = 3 & & & & & & \\
. & . & . & . & . & & \\
. & . & . & . & . & & \\
. & . & . & . & . & & 
\end{array}$$

L  
1  
9  
6  
0

For  $k_x$  equal zero, this result corresponds to that obtained in reference 2 for buckling of a simply supported cylinder subjected to an arbitrary distribution of circumferential stress. Reference 2 is based on Donnell's eighth-order differential equation.

Equation (A10) may be solved for  $k$  for given values of  $k_x$  if the  $s_n$  values are known. By using an increasing number of terms and equations, the solution will converge. The solution for  $k$  should be minimized with respect to  $\beta$  to give the lowest buckling stress. The values of  $\beta$  should be restricted so that an integral number of waves appear around the cylinder circumference. However, this number is usually large enough that  $\beta$  may be thought of as a continuous function.

#### Clamped Ends

An expression for  $w$  which satisfies the condition of clamped ends is

$$w = \sin \frac{\pi y}{\lambda} \sum_{m=1}^{\infty} a_m \left[ \cos(m-1) \frac{\pi x}{L} - \cos(m+1) \frac{\pi x}{L} \right] \quad (A11)$$

and  $V_m$  is therefore

$$V_m = \sin \frac{\pi y}{\lambda} \left[ \cos(m-1) \frac{\pi x}{L} - \cos(m+1) \frac{\pi x}{L} \right] \quad (A12)$$

Performing a similar operation to that for the simply supported cylinder yields

$$\begin{aligned}
 & a_m \left[ M_m (1 + \delta_{1m}) + M_{m+2} \right] - a_{m-2} M_m (1 - \delta_{1m}) (1 - \delta_{2m}) - a_{m+2} M_{m+2} \\
 & + \frac{\beta^2 k}{2} \sum_{i=1}^{\infty} a_i (s_{m-i-2} - 2s_{m-i} + s_{m-i+2} - s_{m+i-2} + 2s_{m+i} - s_{m+i+2}) = 0 \\
 & (m = 1, 2, 3, \dots) \quad (A13)
 \end{aligned}$$

where

$$s_{-1} = s_1$$

Setting the determinant of the coefficients of equations (A13) equal to zero yields the buckling stress

$$\begin{array}{ccccccc}
 & a_1 & & a_2 & & a_3 & \dots \\
 m = 1 & \left| \frac{2(M_1 + M_3)}{\beta^2 k} - 3s_0 + 4s_2 - s_4 \right. & & -2s_1 + 3s_3 - s_5 & & \frac{-2M_2}{\beta^2 k} + s_0 - 3s_2 + 3s_4 - s_6 & \dots \\
 m = 2 & & -2s_1 + 3s_3 - s_5 & & \frac{2(M_2 + M_4)}{\beta^2 k} - 2s_0 + s_2 + 2s_4 - s_6 & & -s_1 + 2s_3 - s_7 & \dots \\
 m = 3 & & & \frac{-2M_2}{\beta^2 k} + s_0 - 3s_2 + 3s_4 - s_6 & & -s_1 + 2s_3 - s_7 & & \frac{2(M_3 + M_5)}{\beta^2 k} - 2s_0 + 2s_2 - s_4 + 2s_6 - s_8 & \dots \\
 & \cdot & & \cdot & & \cdot & & \cdot & \\
 & \cdot & & \cdot & & \cdot & & \cdot & \\
 & \cdot & & \cdot & & \cdot & & \cdot &
 \end{array} = 0 \quad (A14)$$

The solution of equation (A14) is similar to that indicated for equation (A10).

## APPENDIX B

STRUCTURAL BEHAVIOR OF RING-STIFFENED CYLINDER LOADED IN  
AXIAL COMPRESSION AND HEATED UNIFORMLY

## Thermal Stress

It is assumed that the skin is at a constant temperature and the rings are at some lower constant temperature, the difference in temperature being  $\Delta T$ . Also there are many rings so that each bay may be assumed to have the same deflection curve which will be symmetrical about the middle of the bay. Deflections will not vary around the circumference and will be given by the solution of the following (see ref. 6):

$$\left. \begin{aligned} D \frac{d^4 w}{dx^4} + \sigma_x t \frac{d^2 w}{dx^2} + \frac{w}{r^2} Et + \frac{\mu \sigma_x t}{r} &= 0 \\ w \Big|_{x=0} &= (\alpha T - \alpha_R T_R) r \\ w \Big|_{x=L} &= 0 \\ \frac{dw}{dx} \Big|_{x=0} &= 0 \\ \frac{dw}{dx} \Big|_{x=L} &= 0 \end{aligned} \right\} \quad (B1)$$

Deflections are measured positive inward from the position an unstressed cylinder would assume without rings and subjected to a uniform temperature  $T$ . The solution for  $w$  is

$$\begin{aligned} w = & \frac{t\pi^2}{12\sqrt{(1-\mu^2)}Z} (\mu k_x + k_T) \left[ \left( \phi \sinh \frac{\theta}{2} \cos \frac{\phi}{2} \right. \right. \\ & + \theta \cosh \frac{\theta}{2} \sin \frac{\phi}{2} \cosh \frac{\theta}{L} \left( x - \frac{L}{2} \right) \cos \frac{\phi}{L} \left( x - \frac{L}{2} \right) - \left( \theta \sinh \frac{\theta}{2} \cos \frac{\phi}{2} \right. \\ & \left. \left. - \phi \cosh \frac{\theta}{2} \sin \frac{\phi}{2} \sinh \frac{\theta}{L} \left( x - \frac{L}{2} \right) \sin \frac{\phi}{L} \left( x - \frac{L}{2} \right) \right] - \frac{\mu \gamma t}{\sqrt{3(1-\mu^2)}} \end{aligned} \quad (B2)$$

where

$$\theta = \sqrt{\beta} Z \sqrt{1 - \gamma}$$

$$\phi = \sqrt{\beta} Z \sqrt{1 + \gamma}$$

$$\gamma = \frac{\sigma_x r}{Et} \sqrt{3(1 - \mu^2)} = \frac{k_x \pi^2}{4\sqrt{3} Z}$$

$$k_T = \frac{\alpha_T - \alpha_R T_R}{\pi^2} 12(1 - \mu^2) \left(\frac{L}{t}\right)^2$$

The deflection  $w$  will cause a circumferential strain  $\frac{w}{r}$ ; thus,  $\sigma_y$  will be given from the stress-strain relations as

$$\sigma_y = \mu \sigma_x + \frac{w}{r} E \quad (B3)$$

In order to apply this result to the buckling analysis, it is necessary to expand  $\sigma_y$  in a Fourier series. This expansion can be done by substituting equation (B2) into equation (B3) and expanding this result in a series recalling the defining relationship given in equations (A3) and (A4). The resulting Fourier coefficients  $s_n$  are given by

$$\left. \begin{aligned} s_n &= \frac{8\theta\phi(\cosh \theta - \cos \phi)(\theta^2 + \phi^2)}{\left[(\theta^2 + \phi^2 + n^2\pi^2)^2 - 4\phi^2 n^2\pi^2\right](\phi \sinh \theta + \theta \sin \phi)} & (n, \text{ even}) \\ s_n &= 0 & (n, \text{ odd}) \end{aligned} \right\} \quad (B4)$$

and  $k$  is identified as

$$k = \mu k_x + k_T \quad (B5)$$

The only stress component which produces a net force over the cylinder length is  $s_0$ ; thus  $\frac{s_0}{2}$  equals  $\frac{\bar{\sigma}_y}{\sigma_y(0)}$ , the ratio of the average circumferential stress to the stress at a ring.

### Buckling

After the thermal stress is determined, the results can be used in the buckling analysis of appendix A to calculate the buckling temperature. Inasmuch as the odd numbered  $s_n$  are zero, each stability determinant (eqs. (A10) and (A14)) can be factored into two determinants, one containing terms involving the odd-numbered deflection coefficients and the other containing terms involving even-numbered deflection coefficients. The two determinants correspond to deflection patterns symmetrical and antisymmetrical about a plane perpendicular to the axis of the cylinder at the midlength point. The two determinants for the simply supported cylinders are

$$\begin{array}{c|cccc}
 & a_1 & a_3 & a_5 & \dots \\
 m=1 & \frac{2M_2}{\beta^2(\mu k_x + k_T)} - s_0 + s_2 & -s_2 + s_4 & -s_4 + s_6 & \dots \\
 m=3 & -s_2 + s_4 & \frac{2M_4}{\beta^2(\mu k_x + k_T)} - s_0 + s_6 & -s_2 + s_8 & \dots \\
 m=5 & -s_4 + s_6 & -s_2 + s_8 & \frac{2M_6}{\beta^2(\mu k_x + k_T)} - s_0 + s_{10} & \dots \\
 \cdot & \cdot & \cdot & \cdot & \\
 \cdot & \cdot & \cdot & \cdot & \\
 \cdot & \cdot & \cdot & \cdot & 
 \end{array} = 0 \quad (B6)$$

$$\begin{array}{c|cccc}
 & a_2 & a_4 & a_6 & \dots \\
 m=2 & \frac{2M_3}{\beta^2(\mu k_x + k_T)} - s_0 + s_4 & -s_2 + s_6 & -s_4 + s_8 & \dots \\
 m=4 & -s_2 + s_6 & \frac{2M_5}{\beta^2(\mu k_x + k_T)} - s_0 + s_8 & -s_2 + s_{10} & \dots \\
 m=6 & -s_4 + s_8 & -s_2 + s_{10} & \frac{2M_7}{\beta^2(\mu k_x + k_T)} - s_0 + s_{12} & \dots \\
 \cdot & \cdot & \cdot & \cdot & \\
 \cdot & \cdot & \cdot & \cdot & \\
 \cdot & \cdot & \cdot & \cdot & 
 \end{array} = 0 \quad (B7)$$



For the clamped cylinder the following two determinants are obtained:

$$\begin{vmatrix}
 a_1 & a_3 & a_5 & \dots \\
 \frac{2(M_1 + M_2)}{\beta^2(\mu k_x + k_T)} - 3s_0 + 4s_2 - s_4 & \frac{-2M_2}{\beta^2(\mu k_x + k_T)} + s_0 - 3s_2 + 3s_4 - s_6 & s_2 - 4s_4 + 3s_6 - s_8 & \dots \\
 \frac{-2M_2}{\beta^2(\mu k_x + k_T)} + s_0 - 3s_2 + 3s_4 - s_6 & \frac{2(M_2 + M_4)}{\beta^2(\mu k_x + k_T)} - 2s_0 + 2s_2 - s_4 + 2s_6 - s_8 & \frac{-2M_4}{\beta^2(\mu k_x + k_T)} + s_0 - 2s_2 + s_4 - s_6 + 2s_8 - s_{10} & \dots \\
 s_2 - 3s_4 + 3s_6 - s_8 & \frac{-2M_4}{\beta^2(\mu k_x + k_T)} + s_0 - 2s_2 + s_4 - s_6 + 2s_8 - s_{10} & \frac{2(M_4 + M_6)}{\beta^2(\mu k_x + k_T)} - 2s_0 + 2s_2 - s_4 + 2s_6 - s_8 + 2s_{10} - s_{12} & \dots
 \end{vmatrix} = 0 \quad (B8)$$

$$\begin{vmatrix}
 a_2 & a_4 & a_6 & \dots \\
 \frac{2(M_2 + M_4)}{\beta^2(\mu k_x + k_T)} - 2s_0 + s_2 + 2s_4 - s_6 & \frac{-2M_4}{\beta^2(\mu k_x + k_T)} + s_0 - 2s_2 + 2s_4 - s_6 & s_2 - 2s_4 + 2s_6 - s_{10} & \dots \\
 \frac{-2M_4}{\beta^2(\mu k_x + k_T)} + s_0 - 2s_2 + 2s_4 - s_6 & \frac{2(M_4 + M_6)}{\beta^2(\mu k_x + k_T)} - 2s_0 + 2s_2 - s_4 + 2s_6 - s_{10} & \frac{-2M_6}{\beta^2(\mu k_x + k_T)} + s_0 - 2s_2 + s_4 - s_6 + 2s_{10} - s_{12} & \dots \\
 s_2 - 2s_4 + 2s_6 - s_{10} & \frac{-2M_6}{\beta^2(\mu k_x + k_T)} + s_0 - 2s_2 + s_4 - s_6 + 2s_{10} - s_{12} & \frac{2(M_6 + M_8)}{\beta^2(\mu k_x + k_T)} - 2s_0 + 2s_2 - s_4 + 2s_6 - s_8 + 2s_{12} - s_{14} & \dots
 \end{vmatrix} = 0 \quad (B9)$$

Values of  $k_T$  calculated by using the various stability determinants revealed a more suitable buckling parameter. The new parameter  $\tau$  is such that the buckling temperature is given in terms of  $r/t$  rather than  $L/t$  as indicated in the following equation:

$$T = \frac{\alpha_R}{\alpha} T_R + \frac{\tau}{\alpha} \frac{t}{r} \quad (B10)$$

The parameter  $\tau$  is similar to the parameter  $\gamma$  as  $k_T$  is similar to  $k_x$ . The parameter  $\tau$  is related to  $k_T$  by

$$\tau = \frac{\pi^2}{12\sqrt{1 - \mu^2}} \frac{k_T}{Z} \quad (B11)$$

Values of  $k_T$  were found to range through several orders of magnitude with increasing values of  $Z$  whereas  $\tau$  shows only a small variation as shown in figures 1 and 2. Hence, results are better shown in terms of  $\tau$  and values are more easily interpolated.

## Nonrigid rings

In order to correct for ring flexibility it will be assumed that ring deflections can be neglected during buckling, but the effect of these deflections will be included in calculating the circumferential thermal stresses. In other words, the ring is assumed to deflect because of the application of  $\sigma_x$  and  $\sigma_y$  stresses but is assumed to be rigid during the buckling process. This correction is accomplished as follows. The first boundary condition on  $w$  in equation (B1) becomes

$$w \Big|_{x=0}^{x=L} = (\alpha T - \alpha_R T_R) r - \bar{\sigma}_y \frac{Lt}{r\psi} \quad (B12)$$

where  $\psi$  is the deflectional stiffness of the ring and  $\bar{\sigma}_y$  is the average circumferential stress in the skin. For a ring that resists radial deflection by stretching in the circumferential direction,  $\psi$  is given by

$$\psi = \frac{A_R E_R}{r^2} \quad (B13)$$

With the boundary condition (B12) and repeating the thermal stress analysis as before, it is found that  $k$  can be taken as

$$k = \mu k_x + k_T' - \frac{12(1 - \mu^2)}{\pi^2 E} \left(\frac{L}{t}\right)^2 \bar{\sigma}_y \frac{ELt}{r^2 \psi} \quad (B14)$$

with  $s_n$  defined as before and  $k_T'$  the buckling temperature coefficient for nonrigid rings. However,  $\bar{\sigma}_y$  is related to  $k$  and  $s_0$  by

$$\bar{\sigma}_y = k \frac{s_0}{2} \frac{\pi^2 E}{12(1 - \mu^2)} \left(\frac{t}{L}\right)^2 \quad (B15)$$

and  $k_T'$  is therefore given by

$$k_T' = k \left( 1 + \frac{s_0}{2} \frac{ELt}{r^2 \psi} \right) - \mu k_x \quad (B16)$$

This equation can be rewritten in terms of  $\tau'$ ,  $\tau$ , and  $\gamma$ , and with the ring stiffness  $\psi$  given by equation (B13), equation (B16) appears as

$$\tau' = \tau + \left[ \frac{\mu\gamma}{\sqrt{3(1 - \mu^2)}} + \tau \right] \frac{\bar{\sigma}_y}{\sigma_{y(0)}} \frac{ELt}{E_R A_R} \quad (B17)$$

Thus, the buckling temperature coefficient for cylinders with flexible rings equals the coefficient for rigid rings plus an additional term which is proportional to the average circumferential stress in the cylinder.

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## REFERENCES

1. Pride, Richard A., Hall, John B., Jr., and Anderson, Melvin S.: Effects of Rapid Heating on Strength of Airframe Components. NACA TN 4051, 1957.
2. Hoff, N. J.: Buckling of Thin Cylindrical Shell Under Hoop Stresses Varying in Axial Direction. Jour. Appl. Mech., vol. 24, no. 3, Sept. 1957, pp. 405-412.
3. Batdorf, S. B.: A Simplified Method of Elastic-Stability Analysis for Thin Cylindrical Shells. NACA Rep. 874, 1947. (Formerly included in NACA TN's 1341 and 1342.)
4. Peterson, James P.: Correlation of the Buckling Strength of Pressurized Cylinders in Compression or Bending With Structural Parameters. NASA TN D-526, 1960.
5. Seide, Paul, and Weingarten, Victor I.: On the Buckling of Circular Cylindrical Shells Under Pure Bending. Rep. No. EM 9-11, TR-59-0000-00688, Space Tech. Labs., Inc., June 9, 1959.
6. Timoshenko, S.: Theory of Elastic Stability. McGraw-Hill Book Co., Inc., 1936.

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TABLE I.- BUCKLING COEFFICIENTS FOR RING-STIFFENED CYLINDERS  
SUBJECT TO AXIAL COMPRESSION AND THERMAL STRESS

Simple support					Clamped				
$\gamma$	$\tau$	$\beta$	Mode	Order of determinant	$\gamma$	$\tau$	$\beta$	Mode	Order of determinant
<b>Z = 1</b>									
0	3.560	1.00	Symmetrical	4	0	6.178	1.50	Symmetrical	6
.2	3.402	1.00	Symmetrical	3	.2	6.084	1.50	Symmetrical	5
.4	3.234	.95	Symmetrical	3	.4	5.986	1.50	Symmetrical	5
.6	3.054	.90	Symmetrical	3	.6	5.890	1.50	Symmetrical	5
.7	2.960	.90	Symmetrical	3	.7	5.840	1.45	Symmetrical	5
.8	2.857	.85	Symmetrical	4	.8	5.790	1.45	Symmetrical	5
.9	2.752	.80	Symmetrical	3	.9	5.740	1.45	Symmetrical	5
.95	2.695	.80	Symmetrical	3	.95	5.724	1.45	Symmetrical	5
1.20	2.581	.70	Symmetrical	3	2.0	5.171	1.35	Symmetrical	6
1.40	2.053	.60	Symmetrical	3	3.0	4.617	1.25	Symmetrical	5
1.50	1.824	.50	Symmetrical	3	4.0	4.028	1.15	Symmetrical	5
					4.5	3.728	1.05	Symmetrical	5
					5.0	3.531	.90	Symmetrical	5
					5.5	4.501	.70	Symmetrical	5
<b>Z = 3</b>									
0	1.481	1.10	Symmetrical	3	0	2.507	1.50	Symmetrical	5
.2	1.343	1.05	Symmetrical	3	.2	2.445	1.50	Symmetrical	5
.4	1.180	.95	Symmetrical	3	.4	2.394	1.50	Symmetrical	5
.6	.9783	.80	Symmetrical	3	.6	2.347	1.40	Symmetrical	5
.7	.8478	.75	Symmetrical	3	.7	2.350	1.40	Symmetrical	5
.8	.6828	.65	Symmetrical	3	.8	2.301	1.40	Symmetrical	5
.9	.4455	.55	Symmetrical	3	.9	2.316	1.30	Symmetrical	5
.95	.2581	.45	Symmetrical	3	.95	2.316	1.30	Symmetrical	5
<b>Z = 6</b>									
0	1.252	1.25	Symmetrical	3	0	2.068	1.60	Symmetrical	6
.2	1.192	1.15	Symmetrical	3	.2	2.186	1.50	Symmetrical	6
.4	1.140	1.05	Symmetrical	3	.4	2.457	1.45	Symmetrical	6
.6	1.108	1.00	Symmetrical	3	.6	3.194	1.40	Symmetrical	6
.7	1.118	.90	Symmetrical	3	.7	4.144	1.40	Symmetrical	6
.8	1.189	.85	Symmetrical	3	.8	6.282	1.70	Symmetrical	6
.9	1.582	.75	Symmetrical	4	.9	8.487	2.65	Symmetrical	5
.95	2.867	.75	Symmetrical	3	.95	9.356	2.90	Symmetrical	5
					1.05	11.20	3.25	Symmetrical	5
					1.10	12.13	3.38	Antisymmetrical	5
					1.15	13.40	3.60	Antisymmetrical	5
<b>Z = 8</b>									
0	1.354	1.35	Symmetrical	3	0	2.260	1.60	Symmetrical	5
.2	1.390	1.25	Symmetrical	3	.2	2.643	1.60	Symmetrical	5
.4	1.546	1.20	Symmetrical	3	.4	3.669	1.60	Symmetrical	5
.6	2.196	1.10	Symmetrical	3	.6	6.155	2.50	Symmetrical	5
.7	3.193	1.25	Symmetrical	3	.7	7.018	2.80	Antisymmetrical	5
.8	3.782	1.60	Antisymmetrical	3	.8	7.954	3.00	Antisymmetrical	5
.9	4.284	1.60	Antisymmetrical	3	.9	9.080	3.30	Antisymmetrical	5
.95	4.670	1.60	Antisymmetrical	3	.95	9.717	3.50	Antisymmetrical	5
<b>Z = 10</b>									
0	1.511	1.45	Symmetrical	3	0	2.587	1.72	Symmetrical	6
.2	1.670	1.35	Symmetrical	4	.2	3.369	1.72	Symmetrical	6
.4	2.111	1.35	Symmetrical	3	.4	4.961	2.35	Symmetrical	6
.6	3.068	1.80	Antisymmetrical	3	.6	6.080	2.90	Antisymmetrical	6
.7	3.196	1.70	Antisymmetrical	3	.7	6.659	3.10	Antisymmetrical	6
.8	3.401	1.60	Antisymmetrical	3	.8	7.287	3.32	Antisymmetrical	6
.9	3.746	1.55	Antisymmetrical	4	.9	8.034	3.55	Antisymmetrical	6
.95	3.997	1.60	Antisymmetrical	3	.95	8.424	3.68	Antisymmetrical	6
<b>Z = 17.9</b>									
0	1.981	1.85	Symmetrical	4	0	3.542	2.55	Symmetrical	7
.2	2.152	1.95	Symmetrical	5	.2	3.965	3.00	Symmetrical	7
.4	2.236	2.20	Antisymmetrical	4	.4	4.227	3.10	Antisymmetrical	7
.6	2.086	1.90	Antisymmetrical	4	.6	4.502	3.25	Antisymmetrical	7
.7	1.957	1.80	Antisymmetrical	4	.7	4.625	3.35	Antisymmetrical	7
.8	1.754	1.60	Antisymmetrical	4	.8	4.714	3.40	Antisymmetrical	7
.9	1.569	1.60	Antisymmetrical	5	.9	4.751	3.45	Antisymmetrical	7
.95	.9706	1.60	Antisymmetrical	4	.95	4.736	3.45	Antisymmetrical	7
<b>Z = 25</b>									
0	2.052	2.30	Symmetrical	5	0	3.469	3.20	Symmetrical	9
.2	2.082	2.20	Symmetrical	5	.2	3.669	3.40	Antisymmetrical	9
.4	2.006	2.40	Antisymmetrical	5	.4	3.876	3.40	Antisymmetrical	9
.6	1.841	2.10	Antisymmetrical	5	.6	4.142	3.50	Antisymmetrical	9
.7	1.719	2.00	Antisymmetrical	5	.7	4.304	3.60	Antisymmetrical	9
.8	1.550	1.85	Antisymmetrical	5	.8	4.504	3.70	Antisymmetrical	9
.9	1.289	1.60	Antisymmetrical	5	.9	4.783	3.90	Antisymmetrical	9
.95	1.089	1.60	Antisymmetrical	5	.95	4.980	4.00	Antisymmetrical	9
<b>Z = 35</b>									
0	2.063	2.80	Symmetrical	5	0	3.407	3.80	Antisymmetrical	9
.2	2.044	2.80	Symmetrical	5	.2	3.611	3.80	Antisymmetrical	9
.4	1.990	2.70	Antisymmetrical	5	.4	3.919	4.00	Antisymmetrical	9
.6	1.969	2.50	Antisymmetrical	5	.6	4.429	4.30	Antisymmetrical	9
.7	2.010	2.40	Antisymmetrical	5	.7	4.833	4.60	Antisymmetrical	9
.8	2.153	2.30	Symmetrical	5	.8	5.442	5.00	Antisymmetrical	9
.9	1.533	1.70	Symmetrical	5	.9	2.340	1.90	Symmetrical	9
.95	.2430	1.60	Symmetrical	5	.95	.6210	1.70	Symmetrical	9

TABLE I.- BUCKLING COEFFICIENTS FOR RING-STIFFENED CYLINDERS  
SUBJECT TO AXIAL COMPRESSION AND THERMAL STRESS - Concluded

Simple support					Clamped				
$\gamma$	$\tau$	$\beta$	Mode	Order of determinant	$\gamma$	$\tau$	$\beta$	Mode	Order of determinant
$Z = 50$									
0	2.081	3.50	Symmetrical	5	0	3.416	4.40	Antisymmetrical	9
.2	2.057	3.40	Antisymmetrical	5	.2	3.668	4.60	Antisymmetrical	9
.4	2.050	3.30	Antisymmetrical	5	.4	4.040	5.00	Antisymmetrical	9
.6	2.093	3.10	Antisymmetrical	5	.6	4.601	5.40	Antisymmetrical	9
.7	2.138	3.00	Symmetrical	5	.7	3.385	2.40	Symmetrical	9
.8	.9556	2.10	Symmetrical	5	.8	1.136	2.20	Symmetrical	9
.9	.1059	1.90	Symmetrical	5	.9	.2614	2.00	Symmetrical	9
.95	-.07848	1.80	Symmetrical	5	.95	.03571	2.00	Symmetrical	9
$Z = 60$									
0	2.085	3.90	Antisymmetrical	5	0	3.430	4.90	Antisymmetrical	9
.2	2.063	3.70	Antisymmetrical	5	.2	3.687	5.10	Antisymmetrical	9
.4	2.062	3.70	Antisymmetrical	5	.4	4.052	5.40	Antisymmetrical	9
.6	2.076	3.30	Antisymmetrical	5	.6	4.574	5.80	Antisymmetrical	9
.7	2.063	3.10	Symmetrical	5	.7	2.291	2.50	Symmetrical	9
.8	.7663	2.20	Symmetrical	5	.8	.9251	2.40	Symmetrical	9
.9	.1141	2.00	Symmetrical	5	.9	.2430	2.30	Symmetrical	9
.95	-.06140	2.00	Symmetrical	5	.95	.03858	2.20	Symmetrical	9
$Z = 71.5$									
0	2.087	4.10	Antisymmetrical	5	0	3.454	5.30	Antisymmetrical	10
.2	2.067	4.10	Antisymmetrical	5	.2	3.713	5.50	Antisymmetrical	10
.4	2.056	3.90	Antisymmetrical	5	.4	4.061	5.80	Antisymmetrical	9
.6	2.061	3.60	Antisymmetrical	5	.6	4.547	2.80	Symmetrical	8
.7	2.061	3.40	Symmetrical	5	.7	2.350	2.65	Symmetrical	6
.8	.9392	2.30	Symmetrical	4	.8	1.126	2.55	Symmetrical	6
.9	.2502	2.20	Symmetrical	4	.9	.4212	2.45	Symmetrical	6
.95	.02966	2.10	Symmetrical	4	.95	.1815	2.40	Symmetrical	6
$Z = 150$									
0	2.086	6.00	Antisymmetrical	9	0	3.512	7.60	Antisymmetrical	10
.2	2.066	6.00	Antisymmetrical	9	.2	3.794	7.80	Antisymmetrical	10
.4	2.053	5.80	Antisymmetrical	9	.4	4.196	8.20	Antisymmetrical	10
.6	2.057	5.40	Antisymmetrical	9	.6	4.811	8.80	Antisymmetrical	10
.7	2.073	5.20	Antisymmetrical	9	.7	3.920	4.00	Antisymmetrical	10
.8	1.523	3.60	Antisymmetrical	9	.8	1.644	3.80	Antisymmetrical	10
.9	.3664	3.50	Antisymmetrical	9	.9	.4886	3.60	Antisymmetrical	10
.95	.04316	3.40	Antisymmetrical	9	.95	.1678	3.40	Antisymmetrical	10
$Z = 286$									
0	2.087	8.40	Antisymmetrical	11	0	3.605	10.4	Antisymmetrical	12
.2	2.067	8.20	Antisymmetrical	11	.2	3.925	10.8	Antisymmetrical	12
.4	2.055	7.80	Antisymmetrical	11	.4	4.404	11.4	Antisymmetrical	12
.6	2.061	7.40	Antisymmetrical	11	.6	5.191	11.8	Antisymmetrical	12
.7	2.080	7.00	Antisymmetrical	11	.7	4.223	5.10	Antisymmetrical	7
.8	1.614	4.60	Antisymmetrical	11	.8	1.836	4.90	Antisymmetrical	7
.9	.3596	4.20	Antisymmetrical	9	.9	.4863	5.00	Symmetrical	7
.95	.006545	4.20	Antisymmetrical	9	.95	.07847	4.80	Symmetrical	7
$Z = 540$									
0	2.100	11.5	Antisymmetrical	12	0	-----	-----	Antisymmetrical	13
.2	2.085	11.2	Antisymmetrical	12	.2	-----	-----	Antisymmetrical	13
.4	2.085	10.8	Antisymmetrical	12	.4	-----	-----	Antisymmetrical	13
.6	2.112	10.3	Antisymmetrical	12	.6	-----	-----	Antisymmetrical	13
.7	2.146	9.6	Antisymmetrical	12	.7	4.381	7.20	Antisymmetrical	8
.8	1.668	6.60	Symmetrical	12	.8	1.876	7.00	Antisymmetrical	8
.9	.3854	6.20	Symmetrical	12	.9	.5070	6.80	Antisymmetrical	8
.95	.02674	6.40	Antisymmetrical	12	.95	.1047	6.60	Antisymmetrical	8
$Z = 1000$									
0	2.247	15.0	Antisymmetrical	12	0	-----	-----	Antisymmetrical	13
.2	-----	-----	Antisymmetrical	12	.2	-----	-----	Antisymmetrical	13
.4	-----	-----	Antisymmetrical	12	.4	-----	-----	Antisymmetrical	13
.6	-----	-----	Antisymmetrical	12	.6	-----	-----	Antisymmetrical	11
.7	-----	-----	Antisymmetrical	12	.7	4.392	9.75	Antisymmetrical	11
.8	1.697	9.00	Antisymmetrical	12	.8	1.880	9.50	Antisymmetrical	11
.9	.3956	8.70	Antisymmetrical	12	.9	.5130	9.25	Antisymmetrical	11
.95	.03400	8.80	Antisymmetrical	12	.95	.1052	9.25	Antisymmetrical	11

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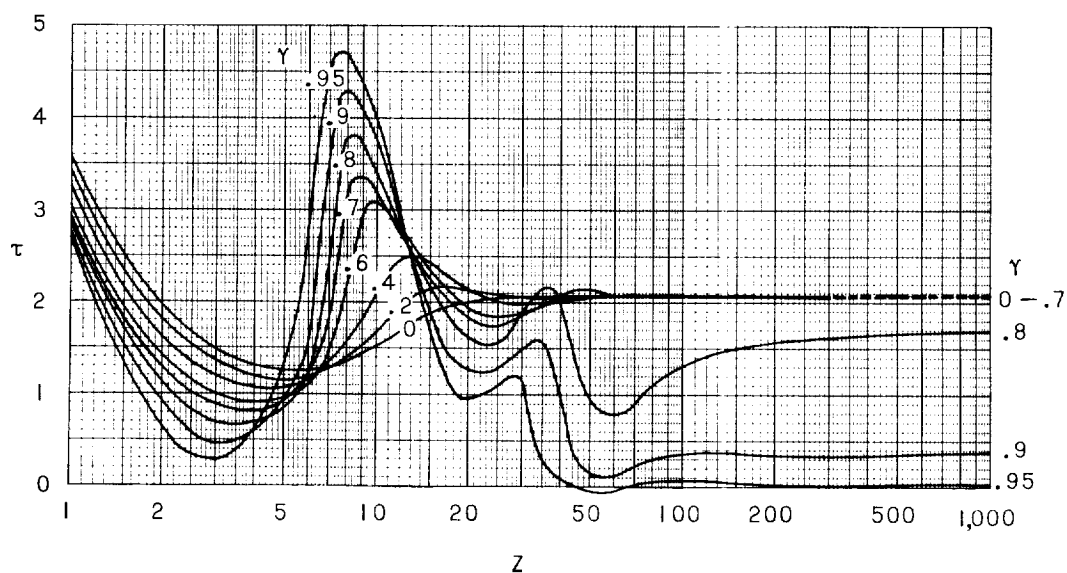


Figure 1.- Temperature buckling coefficient for axially compressed simply supported ring-stiffened cylinders.

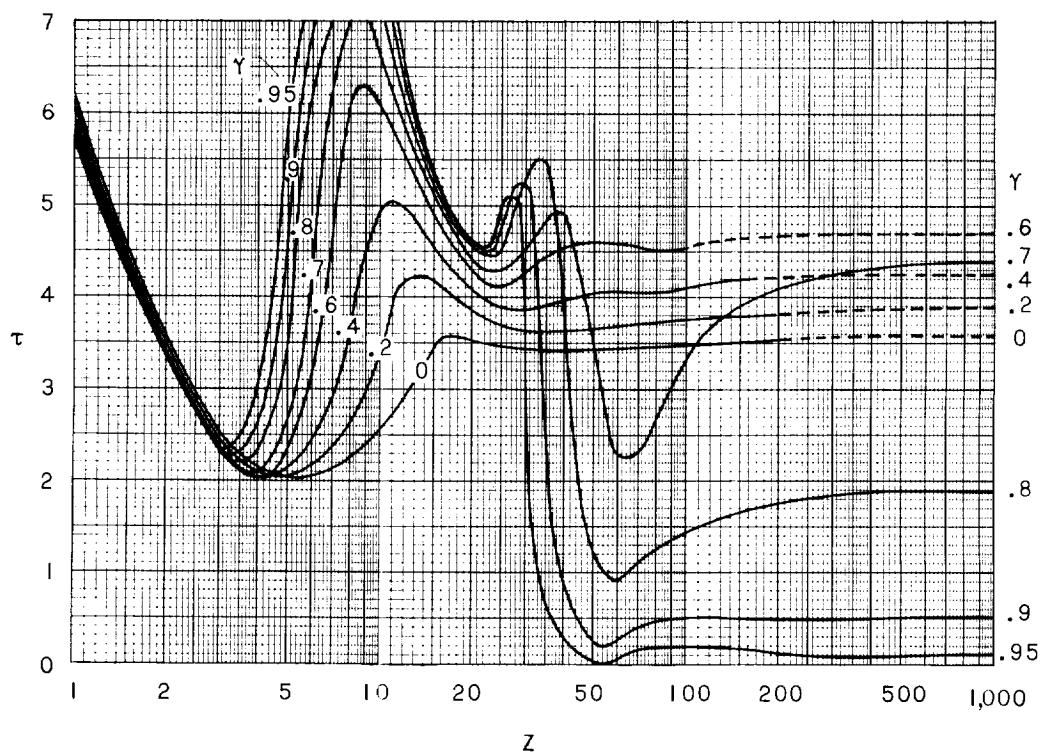


Figure 2.- Temperature buckling coefficient for axially compressed clamped ring-stiffened cylinders.

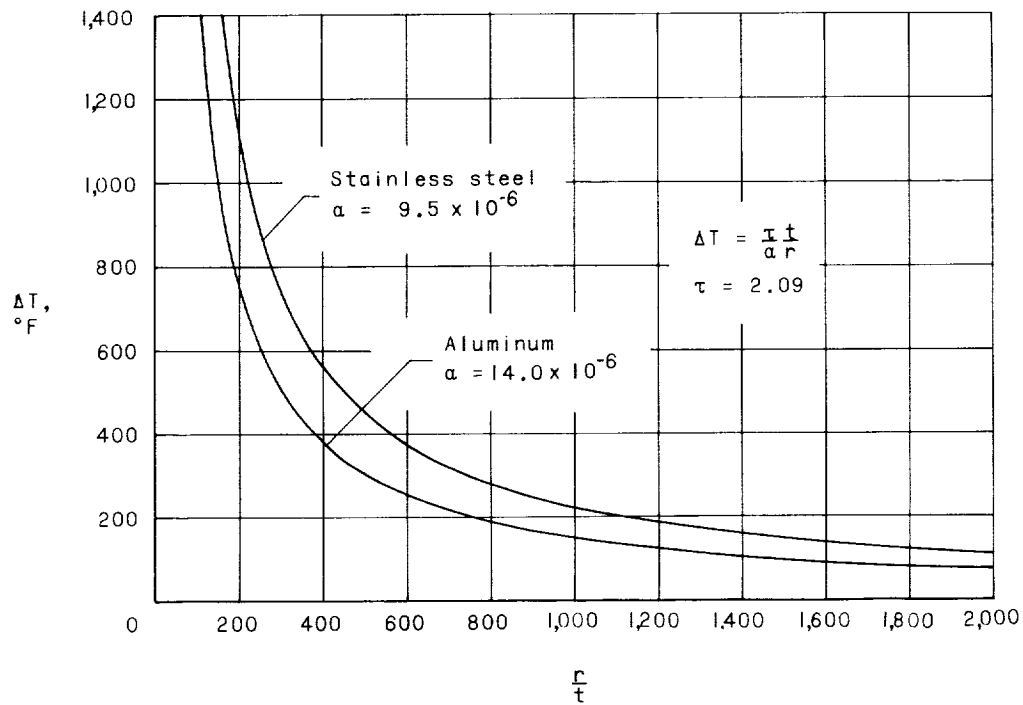


Figure 3.- Temperature increase necessary to cause buckling for simply supported cylinders.  $\gamma < 0.7$ ;  $Z > 60$ .

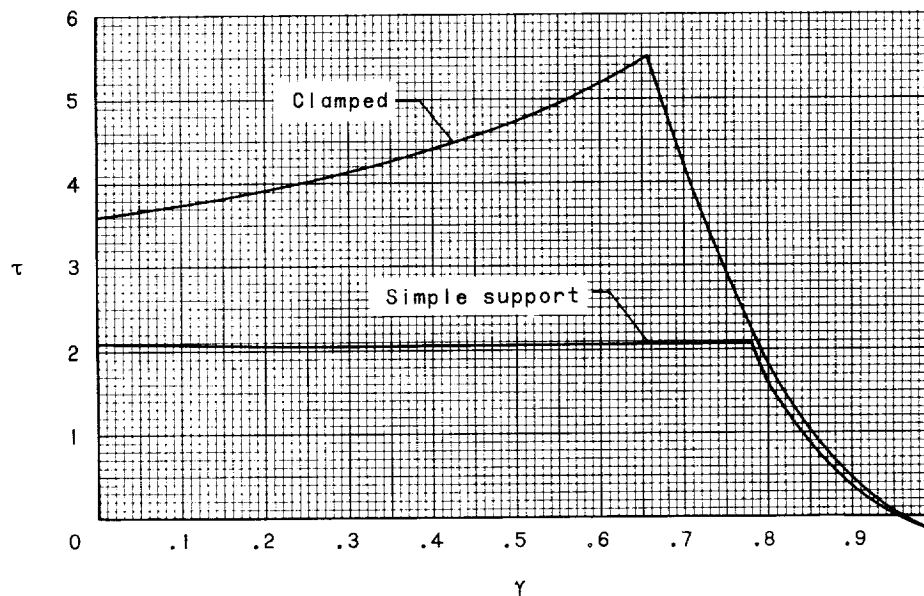


Figure 4.- Variation of  $\tau$  with  $\gamma$  for moderate to large values of  $Z$ .  $Z > 300$ .



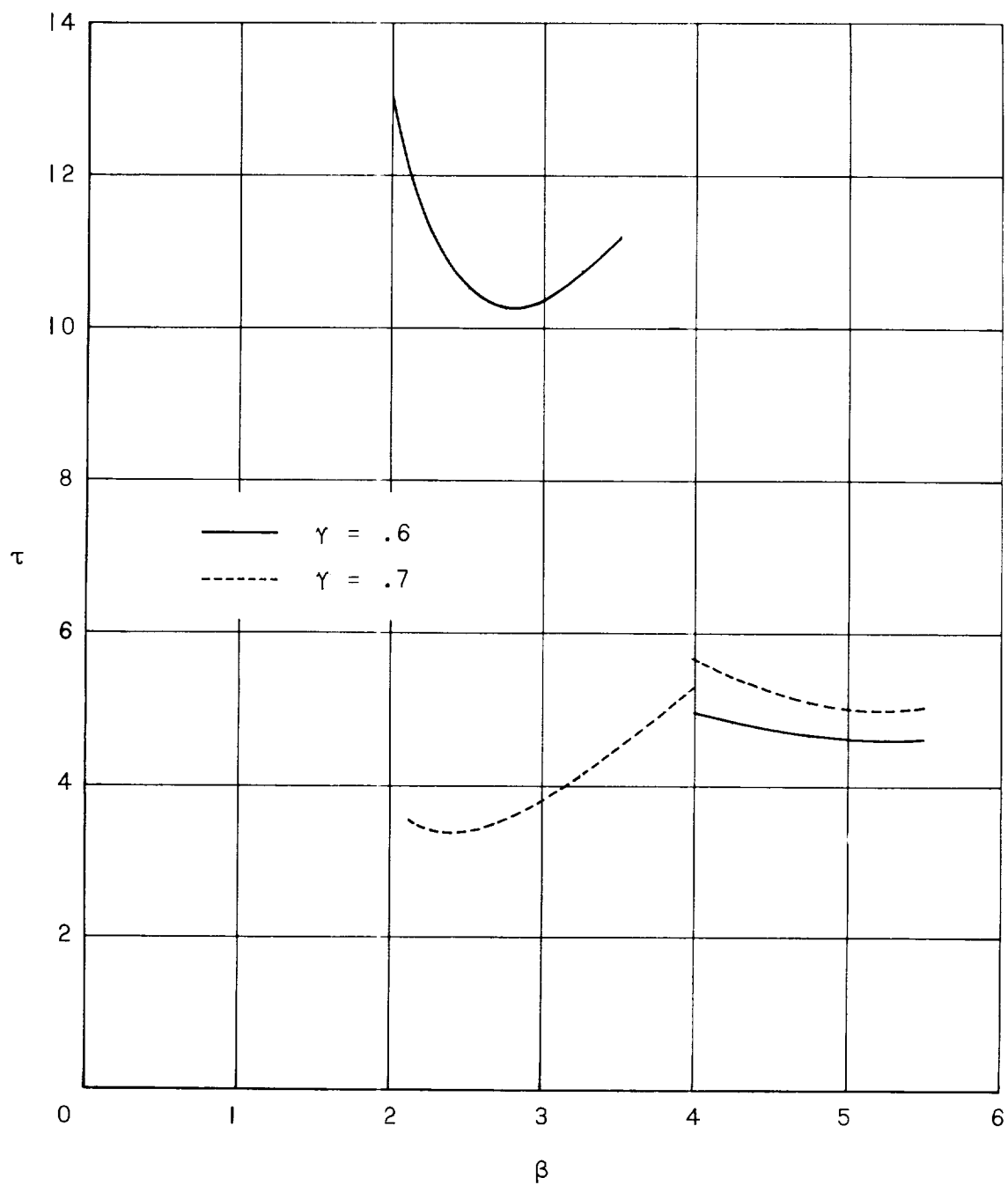


Figure 5.- Plot illustrating change of buckle shape as axial stress is increased.  $\beta = L/\lambda$ .

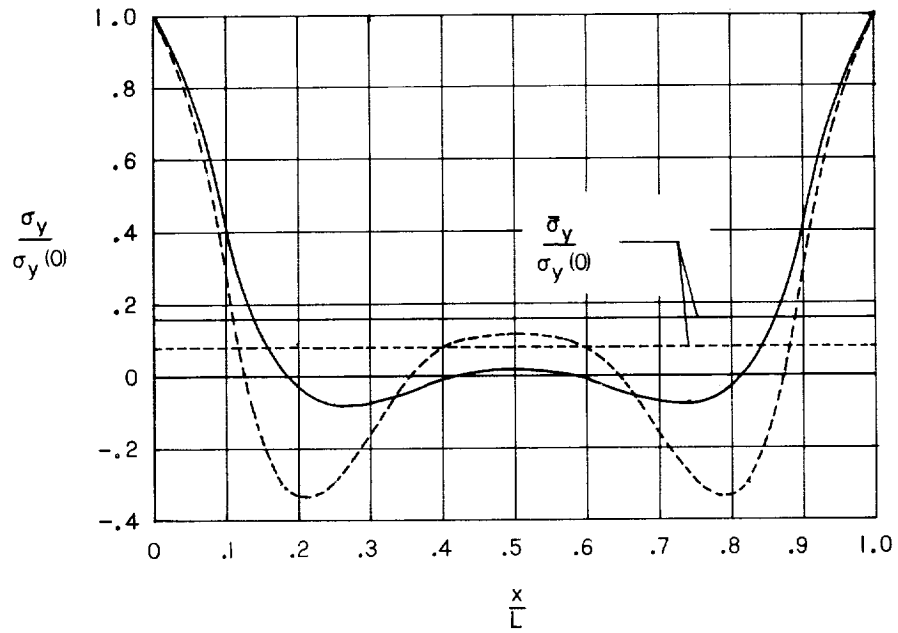
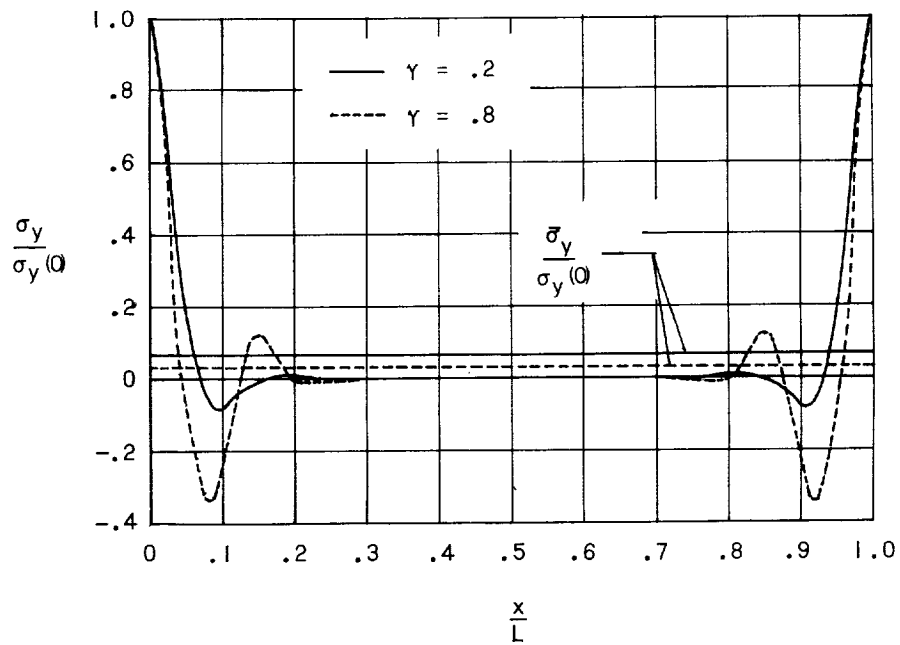
(a)  $Z = 71.5$ .(b)  $Z = 540$ .

Figure 6.- Variation of circumferential stress along cylinder length.

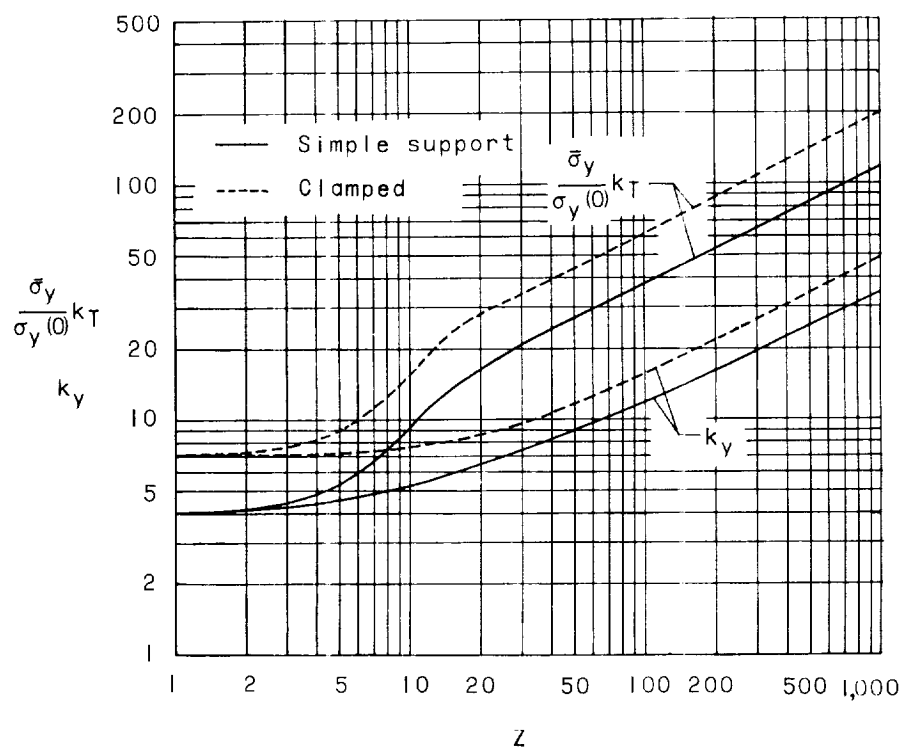


Figure 7.- Comparison of the average thermal buckling stress with the buckling stress for uniform circumferential compression.

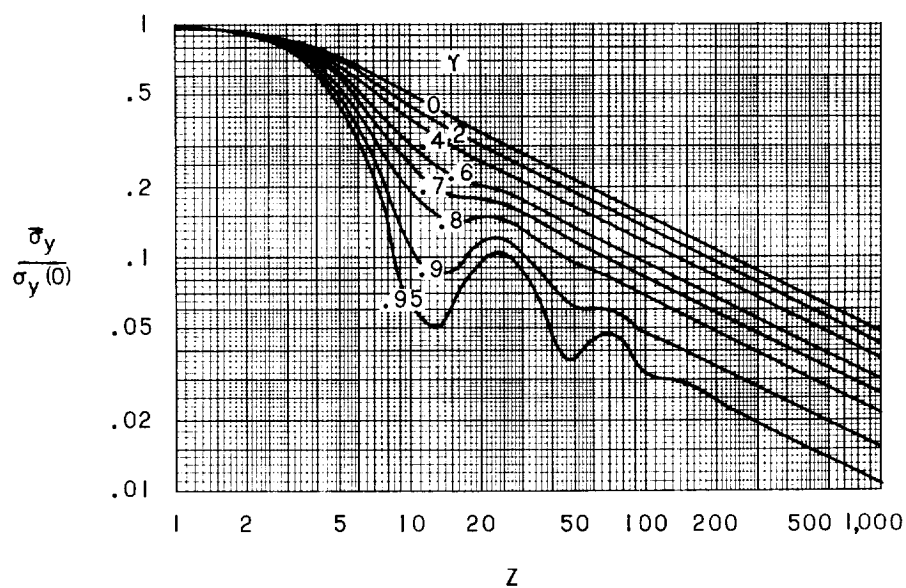


Figure 8.- Ratio of average circumferential stress to stress at a ring.

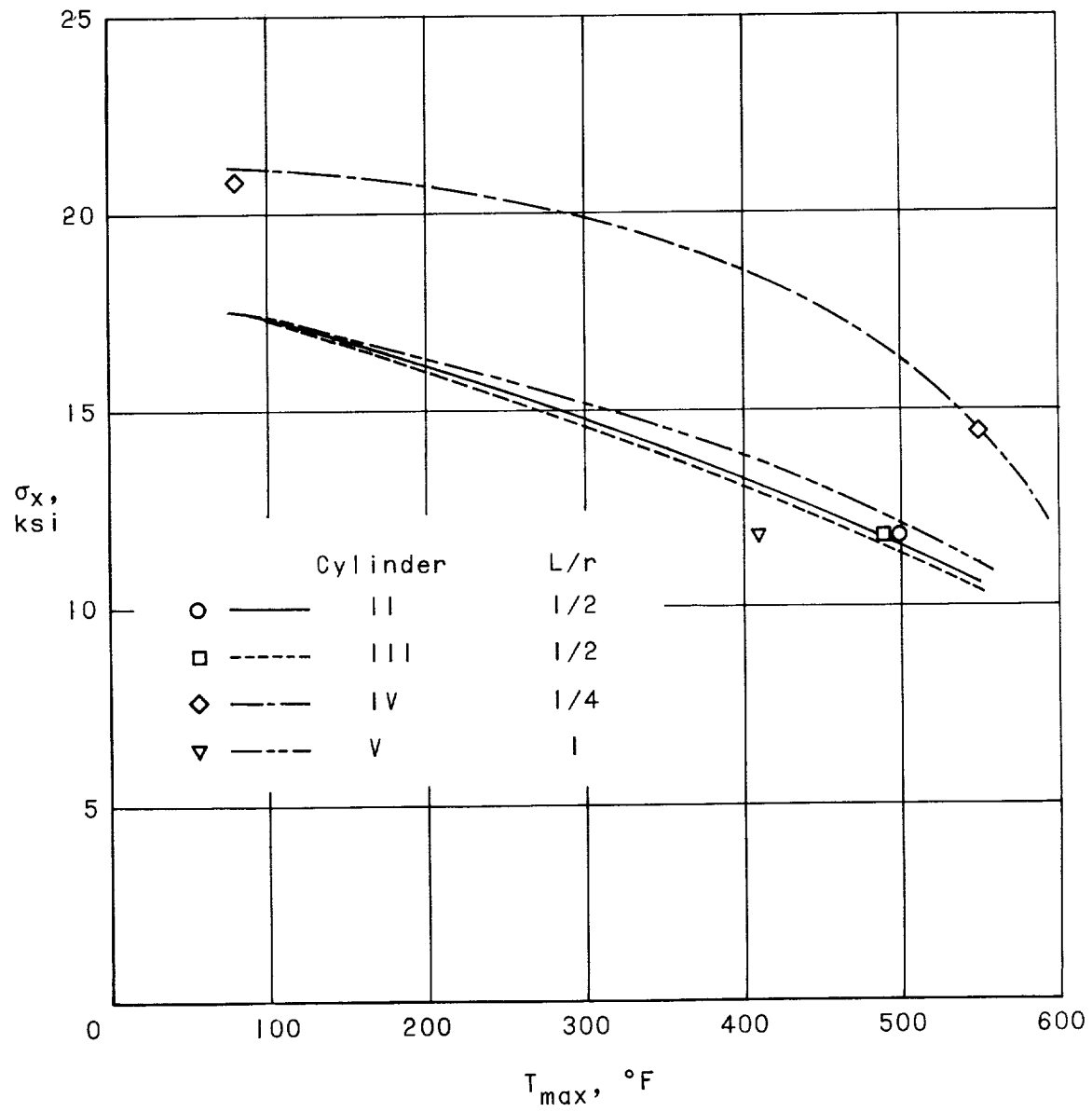


Figure 9.- Comparison of theory with experimental results of reference 1.

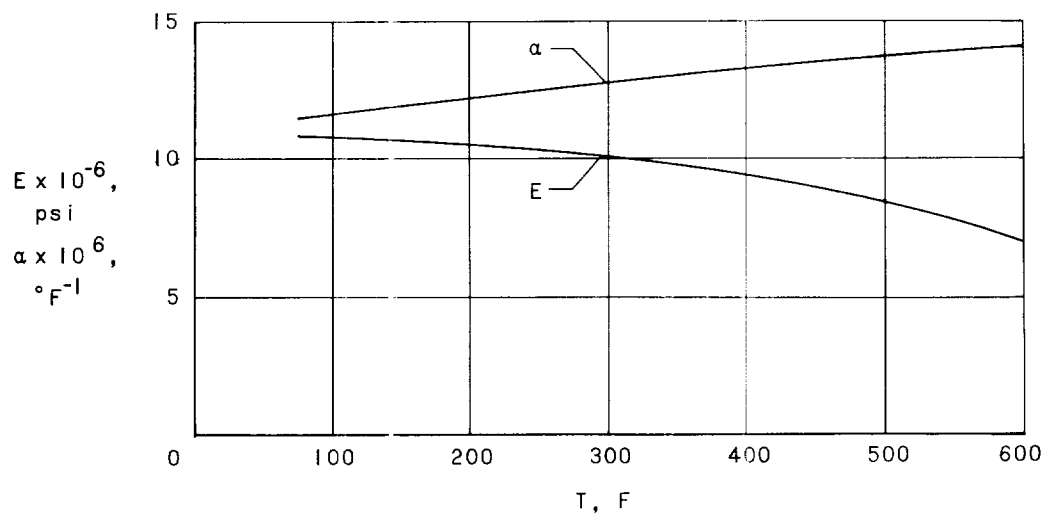


Figure 10.- Variation of  $\alpha$  and  $E$  with temperature for 2024T-3 aluminum alloy.

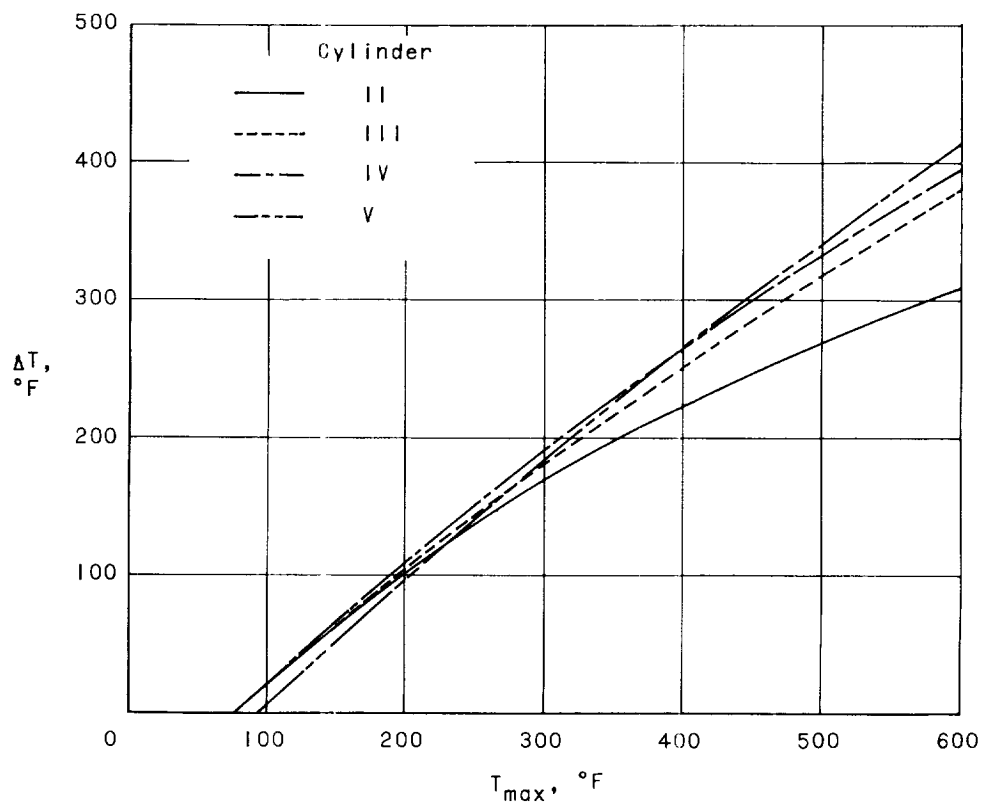


Figure 11.- Variation of average temperature difference between rings and skin with maximum cylinder temperature.

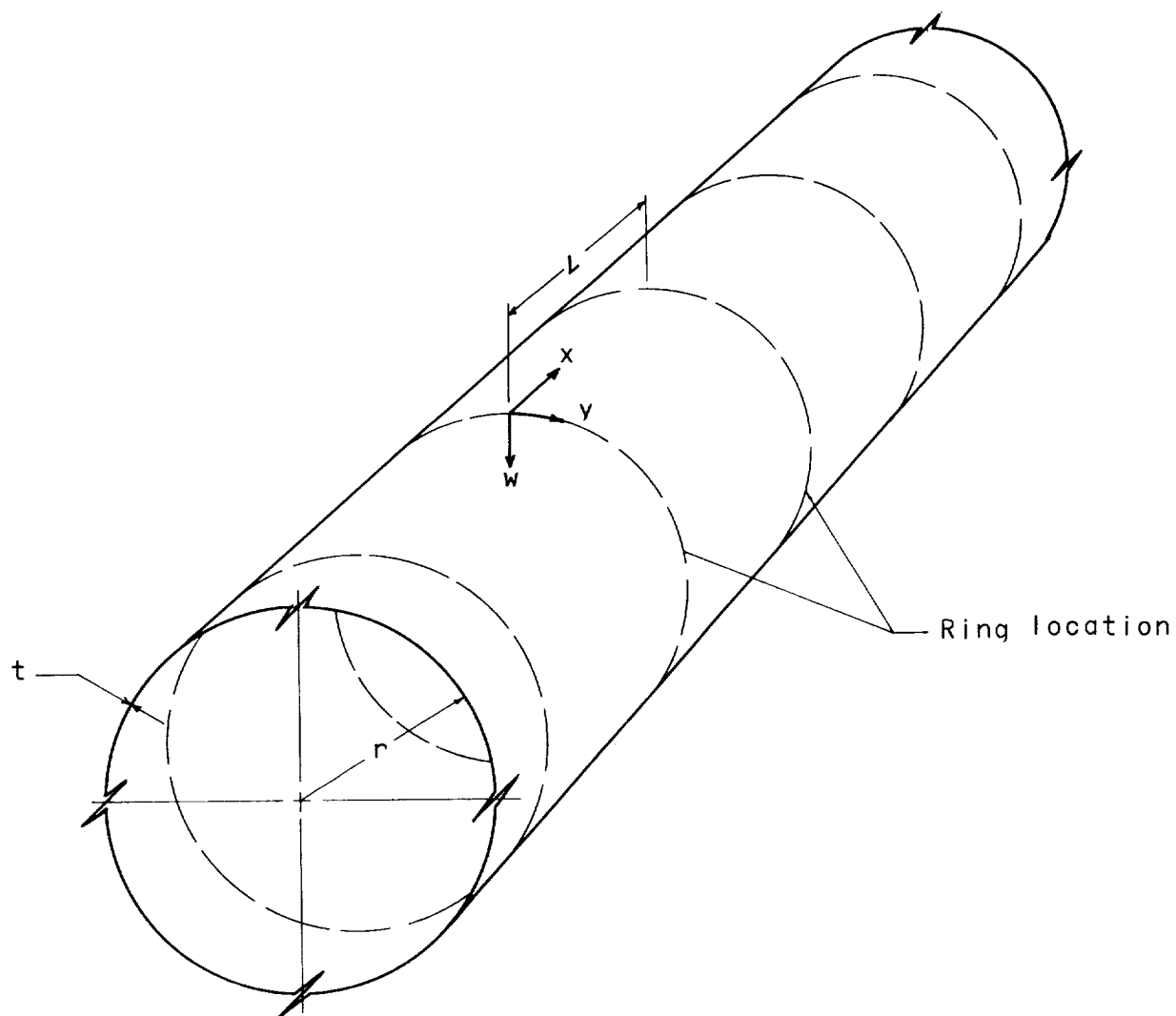


Figure 12.- Cylinder dimensions and coordinate system.